On treating differentials as fractions

In general, it is common in courses not specialized in pure mathematics to perform calculus applications to solve problems in specific areas. A liberty often taken in these courses is to treat differentials as typical quotients which can be separated without any problem. Clearly, there are cases where this does not apply, for example $\left(\frac{dy}{dx}\right)^2 = \frac{(dy)^2}{(dx)^2}$.

Example 1

Consider the following differential equation:

$$2u = \frac{du}{dx} \tag{1}$$

We want to obtain the value of u as a function of x, for this it is common to consider the following:

$$2dx = \frac{du}{u} \tag{2}$$

Integrating both sides with respect to x:

$$2x = \ln(u) + C \tag{3}$$

That is:

$$u = e^{2x-C} \tag{4}$$

This is a correct result, but by treating the differential as a common quotient in order to solve the differential equation, we are somewhat abusing the notation. The problem could have been solved in the following way without the need to treat the differential as a fraction:

$$2u = \frac{du}{dx} \tag{5}$$

$$2 = \frac{du}{dx}\frac{1}{u} \tag{6}$$

By integrating both sides with respect to x, we can see that the result is:

$$2x = \ln(u) + C \tag{7}$$

Since $u = g(x)$ and therefore $\ln(u) = f(g(u))$ and the chain rule tells us that $\ln(u)' = f'g'$. And that's why we know that the integral of $\frac{u'}{u}$ is $\ln(u) + C$.

Example 2

Consider the following example involving the substitution rule in integration, which is essentially an application of the chain rule. Let's consider the integral:

$$\int (2x)e^{(x^2)}\, dx \tag{8}$$

A common approach is to let $u = x^2$, so that by differentiating with respect to x we have $\frac{du}{dx} = 2x$ and then operating as if we were dealing with a fraction: $du = 2xdx$.

Now, let's solve the integral using substitution:

$$\int (2x)e^{(x^2)}\,dx = \int e^{(u)}\,du \tag{9}$$

We change the variables from x to u. By evaluating the right side, we obtain:

$$\int e^{(u)}\,du = e^{(u)} + C = e^{(x^2)} + C \tag{10}$$

Now, let's look at this from the perspective of the chain rule. The chain rule states:

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) \tag{11}$$

In our example, $f(u) = e^{(u)}$ and $g(x) = x^2$, so $f(g(x)) = e^{(x^2)}$. By applying the chain rule:

$$\frac{d}{dx}[e^{(x^2)}] = e^{(x^2)} \cdot 2x \tag{12}$$

When we integrated $(2x)e^{(x^2)}\,dx$, we were essentially finding the antiderivative of $e^{(x^2)} \cdot 2x$ with respect to x. The chain rule tells us that this is $e^{(x^2)}$.

So, although it seemed that we were treating differentials as fractions, this "differentials as fractions" approach can be thought of as a shortcut: it often works, but it is important to understand the underlying reason, which is usually the chain rule.

Example 3

Consider the function

$$z = f(x, y) = x^2 y. \tag{13}$$

Now let's say we want to find the derivative of z with respect to t, where x and y are both functions of t. Specifically,

$$x = g(t) = 3t, \tag{14}$$

$$y = h(t) = t^2. \tag{15}$$

To find $\frac{dz}{dt}$, we can use the chain rule for functions of several variables. The chain rule in this context tells us that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{16}$$

In this particular example,

$$\frac{\partial z}{\partial x} = 2xy, \tag{17}$$

$$\frac{\partial z}{\partial y} = x^2, \tag{18}$$

$$\frac{dx}{dt} = 3, \tag{19}$$

$$\frac{dy}{dt} = 2t. \tag{20}$$

Substituting these values into the chain rule:

$$\frac{dz}{dt} = (2xy)(3) + (x^2)(2t) = 6xy + 2x^2 t. \tag{21}$$

Now, let's suppose for a moment that we can treat differentials as fractions and see what would happen. If we differentiate fully we have

$$dz = z'xdx + z'ydy = 2xy\,dx + x^2\,dy. \tag{22}$$

Treating the differentials as fractions, we could divide both terms by dt and write

$$\frac{dz}{dt} = 2xy\frac{dx}{dt} + x^2\frac{dy}{dt} = 6xy + 2x^2t. \tag{23}$$

Note that this is exactly the same result we obtained using the chain rule. Once again, the method of treating differentials as fractions gave the correct result, but it is actually a manifestation of the chain rule for functions of several variables. It is important to understand that treating differentials as fractions is a form of heuristic, and the correct theoretical basis for these manipulations comes from the chain rule. **Whenever we treat differentials as fractions, we are always fundamentally applying the chain rule.**